ABSTRACT WAVE EQUATIONS WITH ACOUSTIC BOUNDARY CONDITIONS

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Abstract. We define an abstract setting to treat wave equations equipped with time-dependent acoustic boundary conditions on open domains of $\mathbf{R}^n$. We prove a well-posedness result and develop a spectral theory which also allows to prove a conjecture proposed in [12]. Concrete problems are also discussed.

1. Introduction

Wave equations equipped with homogeneous boundary conditions have been studied for a long time. However, other kinds of boundary conditions can also be considered, and for a number of concrete application it seems that the right boundary conditions to impose are time-dependent, cf. [16] and [3].

Certain investigations have in fact led theoreticyal physicists, cf. [20], to investigate wave equations equipped with *acoustic* (or *absorbing*) boundary conditions, which can be written in the form

$$
\text{(ABC)} \qquad \begin{cases} \ddot{\phi}(t,x) = c^2\Delta\phi(t,x), & t\in\mathbf{R},\ x\in\Omega, \\ m(z)\ddot{\delta}(t,z) = -d(z)\dot{\delta}(t,z)-k(z)\delta(t,z)-\rho(z)\dot{\phi}(t,z), & t\in\mathbf{R},\ z\in\partial\Omega, \\ \dot{\delta}(t,z) = \frac{\partial\phi}{\partial\nu}(t,z), & t\in\mathbf{R},\ z\in\partial\Omega. \end{cases}
$$

Here ϕ is the velocity potential of a fluid filling an open domain $\Omega\subset\mathbf{R}^n$, $n\geq 1$; δ is the normal displacement of the (sufficiently smooth) boundary $\partial\Omega$ of Ω; m, d, and k are the mass per unit area, the resistivity, and the spring constant of the boundary, respectively; finally, ρ and c are the unperturbed density of, and the constant speed of sound in the medium, respectively. It is reasonable to assume all these physical quantities to be modelled by essentially bounded functions, with ρ, m real valued and $\inf_{z\in\partial\Omega} m(z)>0$.

Quoting J.T. Beale and S.I. Rosencrans [3] (who denote by G our domain Ω), we point out that 'the physical model giving rise to these conditions is that of a gas undergoing small irrotational perturbations from rest in a domain G with smooth compact boundary', assuming that 'each point of the surface ∂G acts like a spring in response to the excess pressure in the gas, and that there is no transverse tension between neighboring points of ∂G, i.e., the "springs" are independent of each other'.

Operator matrices techniques have been used in this context already in the 1970s, in a series of papers mainly by Beale. The well-posedness of the initial value problem associated with (ABC) has

2000 *Mathematics Subject Classification.* 47D05, 47H20, 35L20.

Key words and phrases. Operator matrices, semigroups of operators, wave equations with acoustic boundary conditions.

This paper has been written while I was a Ph.D. student at the University of Tübingen. I take this occasion to thank the Istituto Nazionale di Alta Matematica "Francesco Severi" for financial support as well as my supervisor, Rainer Nagel, for motivating discussions.

been announced in [3], and a detailed proof has been published shortly afterwards ([2, Thm. 2.1]) under regularity assumptions on the coefficients that are slightly more restrictive than ours.

Recently, acoustic boundary conditions have aroused interest again. For example, C. Gal, G. Goldstein, and J.A. Goldstein have compared them in [12] to more usual dynamical boundary conditions for a wave equation, proving some spectral results and proposing a conjecture that we are now able to prove.

It is remarkable that, back in the 1960s, the Russian school was developing a spectral theory for extremely similar "boundary-contact" problems, whose most peculiar characteristic is that they possibly feature in the boundary conditions differential operators of an order that is higher than those acting in the interior, cf. [16].

In the nice survey [4], B. Belinsky has considered such boundary-contact problems in order to describe some variations on an evocative geophysical model. He has shown that the system of the generalized eigenfunctions of an associated Sturm–Liouville problem forms a Riesz basis on a suitable Sobolev space.

Let us also note that wave equations equipped with simpler boundary conditions but much more complicated coupling relation have been considered by G. Propst and J. Prüss, who proved a well-posedness result in [21, Thm. 4.1].

Our purpose is to present a more general approach to such problems that is based on results on operator matrices with non-diagonal domain mainly obtained by K.-J. Engel (see, e.g., [8], [5] and [15]). This reduces the need for formal computations and allows more general cases, where the conditions of the Lumer–Phillips theorem are harder to check.

Our paper is organized as follows. In Section 2 we introduce the abstract setting we will exploit, and then show the well-posedness of the abstract initial value problem associated with (ABC). In Section 3 we sharpen some known results about Dirichlet operators, and thereby investigate some spectral properties of a certain operator matrix arising in our context. In particular, a conjecture formulated in [12] is proven. In Section 4 we consider a special case where the acoustic boundary conditions shrink to dynamical boundary conditions of first order. This is of indpendent interest, cf. [6]. Finally, motivated by a so-called Timoshenko model discussed in [4, § 3], we prove in Section 5 a well-posedness result for second order problems with neutral acoustic boundary conditions.

2. General setting and well-posedness

Inspired by the setting in [5], we impose the following throughout our paper.

Assumptions 2.1.
(A_1) X, Y, and ∂X are Banach spaces with $Y \hookrightarrow X$.
(A_2) The operator $A : D(A) \to X$ is linear with $D(A) \subset Y$.
(A_3) The operator $R : D(A) \to \partial X$ is linear and surjective.
(A_4) The operators B_1, B_2 are linear and bounded from Y to ∂X.
(A_5) The operators B_3, B_4 are linear and bounded on ∂X.
(A_6) The operator $\binom{A}{R} : D(A) \subset Y \to X \times \partial X$ is closed.
(A_7) The restriction $A_0 := A_{|\ker R}$ generates a cosine operator function with associated phase space $Y \times X$.

Moreover, it will be convenient to define a new operator

$$L := R + B_2, \qquad L : D(A) \to \partial X.$$

We will see that in some applications the operator L is in some sense "more natural" than R. E.g., when we discuss the motivating equation (ABC), the operator B_2 will be the trace operator

and L the normal derivative, while R is a linear combination of the two. This shows that the operator $A_0 = A_{|\ker R}$ can be considered as an abstract version of a operator equipped with Robin boundary conditions. (Recall that, in the context of PDE's, Robin boundary conditions stand for boundary conditions which are a linear combination of Dirichlet and Neumann conditions, cf. [7, § VII.3.2].) The main purpose of this paper is to derive some properties of the wave equation with acoustic boundary conditions from analogous properties of the wave equation with homogeneous Robin (instead of Neumann, as in [2]) boundary conditions.

Remark 2.2. Observe in particular that, due to the boundedness of B_2, the condition (A_6) is satisfied if (and only if) also the operator

$$\begin{pmatrix} A \\ L \end{pmatrix} = \begin{pmatrix} A \\ R \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} : D(A) \subset Y \to X \times \partial X$$

is closed.

Of concern in this paper are abstract second order initial-boundary value problems equipped with (abstract) *acoustic boundary conditions* of the form

$$(\mathrm{AIBVP}_2) \qquad \begin{cases} \ddot{u}(t) &= Au(t), & t \in \mathbf{R}, \\ \ddot{x}(t) &= B_1 u(t) + B_2 \dot{u}(t) + B_3 x(t) + B_4 \dot{x}(t), & t \in \mathbf{R}, \\ \dot{x}(t) &= Lu(t), & t \in \mathbf{R}, \\ u(0) &= f, \qquad \dot{u}(0) = g, \\ x(0) &= h, \qquad \dot{x}(0) = j, \end{cases}$$

on X and ∂X, where the operators A, B_1, B_2, B_3, B_4, and $R = L - B_2$ satisfy the Assumptions 2.1.

We want to recast the second order problem (AIBVP_2) as a first order initial-boundary value problem. Such problems have been thoroughly discussed in [15]. This approach is mostly based on the notion of, and on some results on so-called *one-sided coupled operator matrices*, cf.[8] and [5], and exploits semigroup theory as an essential instrument. In fact, the results in this section are strictly related to properties of the operator matrix with non-diagonal domain $\mathcal{A}$ as defined in (3.1).

Thus, we re-write (AIBVP_2) as a first order abstract initial-boundary value problem

$$(\mathbb{AIBVP}) \qquad \begin{cases} \dot{\mathbb{u}}(t) &= \mathbb{A}\mathbb{u}(t), & t \in \mathbf{R}, \\ \dot{\mathbb{x}}(t) &= \mathbb{B}\mathbb{u}(t) + \tilde{\mathbb{B}}\mathbb{x}(t), & t \in \mathbf{R}, \\ \mathbb{x}(t) &= \mathbb{R}\mathbb{u}(t), & t \in \mathbf{R}, \\ \mathbb{u}(0) &= \mathbb{u}_0, \\ \mathbb{x}(0) &= \mathbb{x}_0, \end{cases}$$

on the Banach spaces

$$\mathbb{X} := Y \times X \times \partial X \qquad \text{and} \qquad \partial\mathbb{X} := \partial X. \tag{2.1}$$

The operator $\mathbb{A}$ on $\mathbb{X}$ is given by

$$\mathbb{A} := \begin{pmatrix} 0 & I_Y & 0 \\ A & 0 & 0 \\ L & 0 & 0 \end{pmatrix}, \qquad D(\mathbb{A}) := D(A) \times Y \times \partial X. \tag{2.2}$$

Further, $\mathbb{R}$ and $\mathbb{B}$ are the operators

$$\mathbb{R} := \begin{pmatrix} R & 0 & 0 \end{pmatrix}, \qquad D(\mathbb{R}) := D(\mathbb{A}), \tag{2.3}$$

and

$$\mathbb{B} := \begin{pmatrix} B_1 + B_4 B_2 & 0 & B_3 \end{pmatrix}, \qquad D(\mathbb{B}) := \mathbb{X}, \tag{2.4}$$

respectively, both from $\mathbb{X}$ to $\partial\mathbb{X}$. Moreover, $\tilde{\mathbb{B}}$ is the operator

$$\tilde{\mathbb{B}} := B_4, \qquad D(\tilde{\mathbb{B}}) := \partial\mathbb{X}, \tag{2.5}$$

on $\partial\mathbb{X}$. Finally, we set the initial data

$$\mathbb{u}_0 := \begin{pmatrix} f \\ g \\ h \end{pmatrix} \qquad \text{and} \qquad \mathbb{x}_0 := j - B_2 f. \tag{2.6}$$

Lemma 2.3 *The following assertions hold.*
- *The restriction* $\mathbb{A}_0 := \mathbb{A}_{|\ker \mathbb{R}}$ *generates a strongly continuous group on* $\mathbb{X}$.
- *The operator* $\mathbb{R}$ *is surjective.*
- *The operator* $\mathbb{B}$ *is bounded from* $\mathbb{X}$ *to* $\partial\mathbb{X}$.
- *The operator* $\tilde{\mathbb{B}}$ *is bounded on* $\partial\mathbb{X}$.
- *The operator* $\binom{\mathbb{A}}{\mathbb{R}} : D(\mathbb{A}) \subset \mathbb{X} \to \partial\mathbb{X}$ *is closed.*

Proof. Observe first that $\ker \mathbb{R} = \{u \in D(A) : Lu = B_2 u\} \times Y \times \partial X$, thus the operator $\mathbb{A}_0$ takes the form

$$\mathbb{A}_0 = \left(\begin{array}{cc|c} 0 & I_Y & 0 \\ A_0 & 0 & 0 \\ \hline B_2 & 0 & 0 \end{array}\right). \tag{2.7}$$

Observe that the perturbation $\begin{pmatrix} B_2 & 0 \end{pmatrix}$ is bounded from $Y \times X$ to ∂X, and the only non-zero diagonal block of $\mathbb{A}_0$ generates by [1, Thm. 3.14.11] a strongly continuous group on $Y \times X$. Therefore, $\mathbb{A}_0$ generates a strongly continuous group on $\mathbb{X}$, and (i) is proven. The remaining claims follow by Assumptions (A_3)–(A_6). $\square$

Therefore, by [15, Prop. 4.1], the following result is immediate.

Proposition 2.4. *The abstract initial-boundary value problem* ($\mathbb{AIBVP}$) *is well-posed in the sense of* [15, § 2].

We now come back to the discussion of the original second order abstract initial-boundary value problem (AIBVP_2).

Definition 2.5. A function $u : \mathbf{R} \to X$ is a classical solution to (AIBVP_2) on $(Y, X, \partial X)$ if
- $u \in C^2(\mathbf{R}, X) \cap C^1(\mathbf{R}, Y)$ and $Lu \in C^1(\mathbf{R}, \partial X)$,
- $u(t) \in D(A)$ for all $t \in \mathbf{R}$,
- u satisfies (AIBVP_2) pointwise.

We will identify solutions to (AIBVP_2) on $(Y, X, \partial X)$ and solutions to ($\mathbb{AIBVP}$) by letting

$$\mathbb{u}(t) \equiv \begin{pmatrix} u(t) \\ \dot{u}(t) \\ h + \int_0^t Lu(s)\, ds \end{pmatrix} \qquad \text{and} \qquad \mathbb{x}(t) \equiv Ru(t), \qquad t \in \mathbf{R}.$$

This is justified by the following.

Lemma 2.6. *The problems* (AIBVP_2) *and* $(\mathbb{AIBVP})$ *are equivalent.*

Proof. Let

$$\mathbb{u} = \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in C^1(\mathbf{R}, \mathbb{X})$$

be a classical solution to $(\mathbb{AIBVP})$ and let $\mathbb{R}\mathbb{u} = Ru = y$. Thus, there holds

$$\begin{cases} \dot{u}(t) &=& v(t), & t \in \mathbf{R}, \\ \dot{v}(t) &=& Au(t), & t \in \mathbf{R}, \\ \dot{x}(t) &=& Lu(t), & t \in \mathbf{R}, \\ \dot{y}(t) &=& (B_1 + B_4 B_2)u(t) + B_3 x(t) + B_4 y(t), & t \in \mathbf{R}, \\ y(t) &=& (L - B_2)u(t), t \in \mathbf{R}, & \\ u(0) &=& f, \qquad v(0) = g, & \\ x(0) &=& h, \qquad y(0) = j - B_2 f, & \end{cases}$$

with $v(t) \in Y$, $t \in \mathbf{R}$, or, equivalently,

$$\begin{cases} \ddot{u}(t) &=& Au(t), & t \in \mathbf{R}, \\ \dot{x}(t) &=& Lu(t), & t \in \mathbf{R}, \\ \ddot{x}(t) &=& B_1 u(t) + B_2 \dot{u}(t) + B_3 x(t) + B_4 \dot{x}(t), & t \in \mathbf{R}, \\ u(0) &=& f, \qquad \dot{u}(0) = g, & \\ x(0) &=& h, \qquad \dot{x}(0) = j, & \end{cases}$$

with $\dot{u}(t) \in Y$, $t \in \mathbf{R}$. To justify this step observe that, by assumption, $u \in C^1(\mathbf{R}, Y)$. Therefore we see that

$$B_2 \dot{u}(t) = B_2 \left(Y\!-\!\lim_{h\to 0} \frac{u(t+h) - u(t)}{h} \right) = \partial X\!-\!\lim_{h\to 0} B_2 \left(\frac{u(t+h) - u(t)}{h} \right) = \frac{d}{dt} B_2 u(t),$$

where we have used the assumption $B_2 \in \mathcal{L}(Y, \partial X)$. Note that this argument does not hold for L.

This shows that u is actually a classical solution to (AIBVP_2) on $(Y, X, \partial X)$. The converse implication follows likewise, and the claim is proven. □

Once we have shown the well-posedness of (AIBVP_2), we can look back at the original initial value problem associated with the wave equation (ABC) introduced in Section 1. Thus, we obtain the following.

Theorem 2.7. *The initial value problem associated with the wave equation with acoustic boundary conditions* (ABC) *on an open domain* $\Omega \subset \mathbf{R}^n$, $n \geq 1$, *with smooth boundary* $\partial\Omega$ *is well-posed. In particular, for all initial data*

$$\phi(0,\cdot) \in H^2(\Omega), \qquad \dot{\phi}(0,\cdot) \in H^1(\Omega), \qquad \delta(0,\cdot) \in L^2(\partial\Omega),$$

$$\text{and} \qquad \dot{\delta}(0,\cdot) \in L^2(\partial\Omega) \qquad \text{such that} \qquad \frac{\partial \phi}{\partial \nu}(0,\cdot) = \dot{\delta}(0,\cdot)$$

there exists a classical solution on $(H^1(\Omega), L^2(\Omega), L^2(\partial\Omega))$ *continuously depending on them.*

Proof. Take first

$$X := L^2(\Omega), \qquad Y := H^1(\Omega), \qquad \partial X := L^2(\partial\Omega).$$

We set

$$A := c^2 \Delta, \qquad D(A) := \left\{ u \in H^{\frac{3}{2}}(\Omega) \,:\, \Delta u \in L^2(\Omega) \right\},$$

$$(Rf)(z) = \frac{\partial f}{\partial \nu}(z) + \frac{\rho(z)}{m(z)} f(z), \qquad f \in D(R) = D(A),\ z \in \partial\Omega,$$

$$B_1 = 0, \qquad (B_2 f)(z) := -\frac{\rho(z)}{m(z)} f(z), \qquad f \in H^1(\Omega),\ z \in \partial\Omega,$$

$$(B_3 g)(z) := -\frac{k(z)}{m(z)} g(z), \qquad (B_4 g)(z) := -\frac{d(z)}{m(z)} g(z), \qquad g \in L^2(\partial\Omega),\ z \in \partial\Omega.$$

By Proposition 2.4 and Lemma 2.6, it suffices to check that the Assumptions (A_1)–(A_7) are satisfied in the above setting.

The Assumptions (A_1) and (A_2) are clearly satisfied. To check the Assumption (A_3), we apply [17, Vol. I, Thm. 2.7.4] and obtain that for all $g \in L^2(\partial\Omega)$ there exists a $u \in H^{\frac{3}{2}}(\Omega)$ such that $\Delta u = 0$ and $\frac{\partial u}{\partial \nu} + \frac{\rho}{m} u_{|\partial\Omega} = g$. The Assumption ($A_4$) holds because the trace operator is bounded from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and because $\frac{\rho}{m} \in L^\infty(\partial\Omega)$, while the Assumption ($A_5$) is satisfied since $\frac{d}{m}, \frac{k}{m} \in L^\infty(\partial\Omega)$.

The Assumption (A_6) is satisfied because the closedness of $\binom{A}{L}$ holds by interior estimates for elliptic operators, (a short proof of this can be found in [5, § 3]), and $B_2 \in \mathcal{L}(Y, \partial X)$, cf. Remark 2.2.

To check Assumption (A_7), observe that the operator $A_0 = A_{|\ker R}$ is in fact (up to the constant c^2) the Laplacian with Robin boundary conditions, that is,

$$A_0 u = c^2 \Delta u, \qquad D(A_0) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + \frac{\rho}{m} u_{|\partial\Omega} = 0 \right\}.$$

This operators is self-adjoint and dissipative up to a scalar perturbation. By the results of [**?**, § 7.1], it generates a cosine operator function with associated phase space $H^1(\Omega) \times L^2(\Omega) = Y \times X$. ☐

Similar well-posedness results for the initial value problem associated with (ABC) have already been obtained in [2]. However, the coefficients ρ, d, k, m are therein assumed to satisfy more restrictive assumptions, as in particular they are supposed to be real and positive (m strictly positive, ρ constant) continuous functions on $\partial\Omega$.

Our Assumptions 2.1 are satisfied by a variety of other operators and spaces. We discuss a biharmonic wave equation with acoustic-type boundary conditions.

Example 2.8. Let $p, q, r, s \in L^\infty(\partial\Omega)$. Then the initial value problem associated with

$$\left\{ \begin{array}{rcll} \ddot{\phi}(t,x) & = & -\Delta^2 \phi(t,x), & t \in \mathbf{R},\ x \in \Omega, \\ \ddot{\delta}(t,z) & = & p(z)\delta(t,z) + q(z)\dot{\delta}(t,z) + r(z)\frac{\partial \phi}{\partial \nu}(t,z) + s(z)\frac{\partial \dot{\phi}}{\partial \nu}(t,z), & t \in \mathbf{R},\ z \in \partial\Omega, \\ \dot{\delta}(t,z) & = & \Delta\phi(t,z), & t \in \mathbf{R},\ z \in \partial\Omega, \\ \phi(t,z) & = & 0, & t \in \mathbf{R},\ z \in \partial\Omega, \end{array} \right.$$

is well-posed. In particular, for all initial data

$$\phi(0,\cdot) \in H^4(\Omega) \cap H^1_0(\Omega), \qquad \dot{\phi}(0,\cdot) \in H^2(\Omega) \cap H^1_0(\Omega), \qquad \delta(0,\cdot) \in L^2(\partial\Omega),$$

$$\text{and} \qquad \dot{\delta}(0,\cdot) \in L^2(\partial\Omega) \qquad \text{such that} \qquad \frac{\partial^2 \phi}{\partial \nu^2}(0,\cdot) = \dot{\delta}(0,\cdot)$$

there exists a classical solution continuously depending on them.

Take

$$X := L^2(\Omega), \qquad Y := H^2(\Omega) \cap H^1_0(\Omega), \qquad \partial X := L^2(\partial\Omega),$$

and consider the operators

$$A := -\Delta^2, \qquad D(A) := \{u \in H^{\frac{5}{2}}(\Omega) \cap H^1_0(\Omega) : \Delta^2 u \in L^2(\Omega)\},$$

$$Ru := (\Delta u)\,|_{\partial\Omega} - s\frac{\partial u}{\partial \nu}, \qquad \text{for all} \quad u \in D(R) := D(A),$$

$$B_1 := r\frac{\partial}{\partial\nu}, \qquad B_2 := s\frac{\partial}{\partial\nu}, \qquad D(B_1) := D(B_2) := Y,$$
$$B_3 x := px, \qquad B_4 := qx, \qquad \text{for all} \quad x \in \partial X.$$

We are only going to prove that A_0, i.e., the restriction of $-\Delta^2$ to

$$D(A_0) := \ker R = \left\{ u \in H^4(\Omega) \cap H^1_0(\Omega) \,:\, (\Delta u)\,|_{\partial\Omega} = s\frac{\partial u}{\partial \nu} \right\},$$

generates a cosine operator function with associated phase space $\left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times L^2(\Omega) = Y \times X$, the remaining Assumptions 2.1 being satisfied trivially.

Take $u, v \in D(A_0)$ and observe that applying the Gauss-Green formulas twice yields

$$\left\langle A_0 u, v \right\rangle_X = -\int_\Omega \Delta^2 u \cdot \overline{v}\, dx = -\int_\Omega \Delta u \cdot \overline{\Delta v}\, dx + \int_{\partial\Omega} s\, \frac{\partial u}{\partial \nu} \cdot \overline{\frac{\partial v}{\partial \nu}}\, d\sigma.$$

It is immediate that A_0 is self-adjoint and dissipative up to a scalar perturbation, hence by the results of [**?**, § 7.1], the generator of a cosine operator function with associated phase space $V \times X$, for some Banach space V. We claim that $V = Y = H^2(\Omega) \cap H^1_0(\Omega)$, thus that the associated phase space is actually $Y \times X$.

Integrating by parts one sees that 0 is not an eigenvalue of $A_0 - \omega$, for $\omega > 0$ large enough, hence $-A_0 + \omega$ is a strictly positive self-adjoint operator. The domain of its square root coincides by [14, Thm. VI.2.23] with the form domain of A_0. Moreover, one can directly check that the form domain of A_0 is $H^2(\Omega) \cap H^1_0(\Omega)$. Therefore, by [9, Prop. VI.3.14] we deduce that the associated phase space of A_0 is $V \times X = [D(-A_0 + \omega)^{\frac{1}{2}}] \times X = \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times L^2(\Omega)$ as claimed.

Remark 2.9. Among further operators and spaces fitting into our abstract framework we list the following. In both cases, the operator B_2 is defined as in the proof of Theorem 2.7.

a) $X := L^2(\Omega),\ Y := H^1(\Omega),\ \partial X := L^2(\partial\Omega)$,
$Au(x) := \nabla(a(x)\nabla u(x)),\ x \in \Omega$, with the function $a \geq 0$ sufficiently regular on $\overline{\Omega}$,
$Lu(z) =< a(z)\nabla u(z), \nu(z) >,\ z \in \partial\Omega$,
for $u \in D(A) := \{H^{\frac{3}{2}}(\Omega) : Au \in L^2(\Omega)\}$.

b) $X := L^2(\Omega),\ Y = H^2(\Omega),\ \partial X := L^2(\partial\Omega)$,
$Au := -\Delta^2 u$,
$Lu := -\frac{\partial \Delta u}{\partial \nu}$,
for $u \in D(A) := \left\{ H^{\frac{7}{2}}(\Omega) : \Delta^2 u \in L^2(\Omega),\ (\Delta u)\,|_{\partial\Omega} = 0 \right\}$.

In either case, A_0 is self-adjoint and dissipative up to a scalar perturbation. This ensures that A_0 generates a cosine operator function.

Before concluding this section, let us emphasize that our proof of the well-posedness of (AIBVP$_2$) actually relies on the reformulation of ($\mathbb{AIBVP}$) as a first order abstract Cauchy problem

$$(\mathcal{ACP}) \qquad \left\{ \begin{array}{rcl} \dot{\mathcal{U}}(t) &=& \mathcal{A}\,\mathcal{U}(t), \quad t \in \mathbf{R}, \\ \mathcal{U}(0) &=& \mathcal{U}_0, \end{array} \right.$$

on $\mathcal{X} := \mathbb{X} \times \partial\mathbb{X}$, where

$$(2.8) \qquad \mathcal{A} := \begin{pmatrix} \mathbb{A} & 0 \\ \mathbb{B} & \tilde{\mathbb{B}} \end{pmatrix}, \qquad D(\mathcal{A}) := \left\{ \begin{pmatrix} \mathbb{u} \\ \mathbb{x} \end{pmatrix} \in D(\mathbb{A}) \times \partial\mathbb{X} \,:\, \mathbb{R}\mathbb{u} = \mathbb{x} \right\},$$

is an operator on $\mathcal{X}$ and

$$\mathcal{U}(t) := \begin{pmatrix} \mathbb{u}(t) \\ \mathbb{R}\mathbb{u}(t) \end{pmatrix}, \quad t \in \mathbf{R}, \qquad \mathcal{U}_0 := \begin{pmatrix} \mathbb{u}_0 \\ \mathbb{x}_0 \end{pmatrix}.$$

In fact, it has beens shown in [15, § 2] that a function $\mathbb{u} : \mathbf{R} \to \mathbb{X}$ is a classical solution to ($\mathbb{AIBVP}$) if it is the first coordinate of a classical solution $\mathcal{U} : \mathbf{R} \to \mathcal{X}$ to ($\mathcal{ACP}$), and in fact the proof of Proposition 2.4 relies on the fact that, by Lemma 2.3, $\mathcal{A}$ generates a group on $\mathcal{X}$, cf. [15, Prop. 4.1].

Remark 2.10. Observe that, in order to show the well-posedness of (a problem equivalent to) ($\mathcal{ACP}$) on an open bounded domain Ω of $\mathbf{R}^3$, Beale introduced the product space

$$(H^1(\Omega;\rho)/\mathbf{C}) \times L^2(\Omega;\frac{\rho}{c^2}) \times L^2(\partial\Omega;k) \times L^2(\partial\Omega;m).$$

Such a space looks somehow artificial, due to the quotient appearing in the first coordinate and to the weights of the remaining L^2-spaces. He then showed that a certain operator matrix (different from our $\mathcal{A}$) verifies the conditions of the Lumer–Phillips theorem – that is, the energy of the solutions to (ABC) is nonincreasing for time $t \geq 0$. Moreover, if the parameter $d \equiv 0$, then also the conditions of Stone's theorem are satisfied – that is, the energy is constant for $t \in \mathbf{R}$. Also, Beale showed that his operator matrix does not have compact resolvent and computed its essential spectrum, but his techniques can hardly be applied to problems on domains of $\mathbf{R}^n$, $n \neq 3$.

Since Beale's paper, the theory of asymptotics for (semi)groups has been widely developped. In particular, it is now known that every bounded strongly continuous group (resp., semigroup) whose generator has compact resolvent is almost periodic (resp., asymptotically almost periodic), cf. [1, Chapt. 5]. (More generally, every bounded strongly continuous semigroup with only countably many spectral values on $i\mathbf{R}$ is asymptotically almost periodic. It seems therefore worthwhile to develop a complete spectral theory for the problem (AIBVP$_2$).) This is the main aim of Section 3.

The main drawback of our own approach is that we fail to produce an energy space for the motivating equation (ABC) on a bounded domain $\Omega \subset \mathbf{R}^n$, i.e., the group generated by $\mathcal{A}$ is not contractive, as it can be seen already in the case of $n = 1$.

However, our approach has other advantages. In particular, the above operator matrix $\mathcal{A}$ can be written as

$$\mathcal{A} := \mathcal{A}_1 + \mathcal{A}_2 := \begin{pmatrix} \mathbb{A} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbb{B} & \tilde{\mathbb{B}} \end{pmatrix}, \tag{2.9}$$

where $\mathcal{A}_1$ is the generator of a strongly continuous group on $\mathcal{X}$ and $\mathcal{A}_2$ is a bounded operator, which is compact if and only if $\dim \partial \mathrm{X} < \infty$. A decomposition of this type cannot be performed on the operator matrix considered by Beale, and has some interesting consequences: Observe that, in the context of (ABC), the group generated by $\mathcal{A}_1$ governs the inital value problem associated with the wave equation with inhomogeneous (static) Robin boundary conditions

$$\text{(iRBC)} \qquad \begin{cases} \ddot{\psi}(t,x) = c^2 \Delta \psi(t,x), & t \in \mathbf{R},\ x \in \Omega, \\ \frac{\partial \psi}{\partial \nu}(t,z) + \frac{\rho(z)}{m(z)} \psi(t,z) = \gamma(z), & t \in \mathbf{R},\ z \in \partial\Omega, \end{cases}$$

where $\gamma(z) := \frac{\partial \psi}{\partial \nu}(0,z) + \frac{\rho(z)}{m(z)} \psi(0,z)$.

The solution ψ to (iRBC) is given by the group generated by $\mathcal{A}_1$ and can be explicitly written down, cf. [15, Thm. 3.6]. Thus, due to (2.9), the solution ϕ to (ABC) can be obtained by the Dyson–Phillips series, cf. [9, Thm. III.1.10]. Further, by [9, Cor. III.1.11] we deduce that

$$\|\phi(t,\cdot) - \psi(t,\cdot)\|_{L^2(\Omega)} \leq tM,$$

for $t \in [0,1]$ and some constant M.

3. Regularity and spectral theory

Due to the important role played by the operator matrix $\mathcal{A}$ defined in (2.8), we are interested in developing a spectral theory for it. To this purpose we are still imposing the Assumptions 2.1.

As a first step, we recall the notion of Dirichlet operators. To obtain an optimal boundedness result, we need the following.

Lemma 3.1. *Let Z be a Banach space such that $Z \hookrightarrow Y$, and consider the operator matrix*

$$\begin{pmatrix} A & 0 \\ R & 0 \end{pmatrix} : D(A) \times \partial X \to X \times \partial X$$

on $X \times \partial X$. Then its part in $Z \times \partial X$ is closed.

Proof. We can consider the part $A_|$ of A in Z and let $\binom{u_n}{x_n}_{n\in\mathbf{N}} \subset D(A_|) \times \partial X$ such that

$$\begin{pmatrix} u_n \\ x_n \end{pmatrix} \to \begin{pmatrix} u \\ x \end{pmatrix} \qquad \text{in } Z \times \partial X$$

and

$$\begin{pmatrix} A_| & 0 \\ R & 0 \end{pmatrix} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \begin{pmatrix} Au_n \\ Ru_n \end{pmatrix} \to \begin{pmatrix} w \\ y \end{pmatrix} \qquad \text{in } Z \times \partial X.$$

Since $Z \hookrightarrow Y$, we can apply the closedness of $\binom{A}{R}$ and conclude that $u \in D(A)$, $Au = w$, and $Ru = y$. This completes the proof. ☐

We are now in the position to show the following refinement of [5, Lemma 2.2].

Lemma 3.2. *If $\lambda \in \rho(A_0)$, then the restriction $R\big|_{\ker(\lambda - A)}$ has an inverse*

$$D_\lambda^{A,R} : \partial X \to \ker(\lambda - A),$$

called Dirichlet operator associated with A and R. Moreover, $D_\lambda^{A,R}$ is bounded from ∂X to Z for every Banach space Z satisfying $D(A^\infty) \subset Z \hookrightarrow Y$.

Proof. The existence of the Dirichlet operator $D_\lambda^{A,R}$ follows from [5, Lemma 2.2], due to the Assumptions (A_3),(A_7).

Observe now that $\ker(\lambda - A) \subset D(A^\infty)$. Therefore the boundedness of $D_\lambda^{A,R}$ from ∂X to some Banach space Z containing $D(A^\infty)$ is equivalent to the closedness of the operator $R\big|_{\ker(\lambda-A)}$: $\ker(\lambda - A) \subset Z \to \partial X$.

To show that $R\big|_{\ker(\lambda-A)}$ is actually closed, take $(u_n)_{n\in\mathbf{N}} \subset \ker(\lambda - A)$ such that $u_n \overset{Z}{\to} u$ and $Ru_n \overset{\partial X}{\to} x$. It follows that $Au_n = \lambda u_n \overset{Z}{\to} \lambda u$, that is

$$\begin{pmatrix} A & 0 \\ R & 0 \end{pmatrix}_| \begin{pmatrix} u_n \\ 0 \end{pmatrix} \to \begin{pmatrix} \lambda u \\ x \end{pmatrix} \qquad \text{in } Z \times \partial X.$$

By Lemma 3.1 we conclude that $u \in D(A)$ and that $Au = \lambda u$, $Ru = x$. ☐

Remarks 3.3. a) The above Dirichlet operators $D_\lambda^{A,R}$ can also be interpreted as follows: Lemma 3.2 says that the abstract eigenvalue problem

$$\begin{cases} Au = \lambda u & \text{in } X, \\ Ru = x & \text{in } \partial X, \end{cases}$$

has a unique solution $D_\lambda^{A,R} x$ for all $x \in \partial X$, and that the dependence on x is continuous.

b) If ∂X is finite dimensional, or else if a Banach space Z as in the statement of Lemma 3.2 can be chosen to be *compactly* embedded in Y, then we obtain that the Dirichlet operators are compact from ∂X to Y.

c) We finally observe that, by definition,

$$LD_\lambda^{A,R} = I_{\partial X} + B_2 D_\lambda^{A,R} \qquad \text{for all } \lambda \in \rho(A_0), \tag{3.1}$$

and therefore $LD_\lambda^{A,R}$ is a bounded operator on ∂X.

We recall that the resolvent set of the operator matrix

$$\begin{pmatrix} 0 & I_Y \\ A_0 & 0 \end{pmatrix} \qquad \text{with domain} \qquad D(A_0) \times Y \tag{3.2}$$

on the space $Y \times X$ is given by $\{\lambda \in \mathbf{C} : \lambda^2 \in \rho(A_0)\}$. Accordingly, we obtain the following.

Lemma 3.4. *The resolvent set of* $\mathbb{A}_0$ *is given by*

$$\rho(\mathbb{A}_0) = \left\{\lambda \in \mathbf{C} : \lambda \neq 0,\ \lambda^2 \in \rho(A_0)\right\}.$$

For $\lambda \in \rho(\mathbb{A}_0)$ *there holds*

$$R(\lambda, \mathbb{A}_0) = \begin{pmatrix} \lambda R(\lambda^2, A_0) & R(\lambda^2, A_0) & 0 \\ A_0 R(\lambda^2, A_0) & \lambda R(\lambda^2, A_0) & 0 \\ -B_2 R(\lambda^2, A_0) & -\frac{1}{\lambda} B_2 R(\lambda^2, A_0) & \frac{1}{\lambda} I_{\partial X} \end{pmatrix}.$$

Proof. The resolvent operator of the operator matrix introduced in (3.2) is given by [1, (3.107)]. Then the claimed formula can be checked directly. □

Lemma 3.5. *For the operator* $(\mathbb{A}, D(\mathbb{A}))$ *defined in (2.2) we obtain*

$$D(\mathbb{A}^{2k-1}) = D(A^k) \times D((A^{k-1})_{|Y}) \times \partial X \qquad \text{and}$$

$$D(\mathbb{A}^{2k}) = D((A^k)_{|Y}) \times D(A^k) \times \partial X \qquad \text{for all } k \in \mathbf{N}.$$

In particular, $D(\mathbb{A}^\infty) = D(A^\infty) \times D(A^\infty) \times \partial X$.

Proof. The claim follows by induction on n, using the fact that

$$D((A^k)_{|Y}) = \{u \in D(A) : Au \in D((A^{k-1})_{|Y})\}$$

and recalling that $D(A) \subset Y$. □

This allows to obtain the following regularity result.

Corollary 3.6. *Assume that the initial data* f, g *are in*

$$\mathcal{D}_0^\infty := \bigcap_{k=0}^\infty \{w \in D(A^k) : RA^k w = LA^k w = 0\}. \tag{3.3}$$

If further $h = j = 0$*, then the solution* $u = u(t)$ *to* (AIBVP$_2$) *is in* $D(A^\infty)$ *for all* $t \in \mathbf{R}$.

Proof. Taking into account Lemma 3.5, the inclusion $(\mathcal{D}_0^\infty)^2 \times \{0\}^2 \subset D(\mathcal{A}^\infty)$ can be proven by induction. Then, one only needs to recall that the group generated by $\mathcal{A}$ maps $D(\mathcal{A}^n)$ in itself for all $n \in \mathbf{N}$, and to observe that $D(\mathcal{A}^\infty) \subset D(A^\infty)^2 \times \partial X^2$. □

Example 3.7. Consider the framework introduced in the proof of Theorem 2.7 to treat the motivating equation (ABC). Corollary 3.6 yields a regularity result that is similar to [2, Thm. 2.2]. In fact, $\mathcal{D}_0^\infty$ defined in (3.3) contains the set of all functions of class $C^\infty(\Omega)$ that vanish in a suitable neighborhood of $\partial\Omega$. On the other hand, $D(A^\infty) \subset H^{\frac{3}{2}}(\Omega) \cap C^\infty(\Omega)$.

Observe now that, by Lemma 2.3, the operators $\mathbb{A}$ and $\mathbb{R}$ satisfy the Assumptions 2.1. Therefore by Lemma 3.2 and for $\lambda \in \rho(\mathbb{A}_0)$, one obtains the existence of the Dirichlet operator $D_\lambda^{\mathbb{A},\mathbb{R}}$ associated with $\mathbb{A}$ and $\mathbb{R}$. More precisely, the following representation holds.

Lemma 3.8. *Let $\lambda \in \rho(\mathbb{A}_0)$. Then the Dirichlet operator $D_\lambda^{\mathbb{A},\mathbb{R}}$ exists and is represented by*

$$D_\lambda^{\mathbb{A},\mathbb{R}} = \begin{pmatrix} D_{\lambda^2}^{A,R} \\ \lambda D_{\lambda^2}^{A,R} \\ \frac{1}{\lambda} L D_{\lambda^2}^{A,R} \end{pmatrix}. \tag{3.4}$$

Moreover, $D_\lambda^{\mathbb{A},\mathbb{R}}$ is bounded from $\partial\mathbb{X}$ to $W \times Z \times \partial X$ for every two Banach spaces W, Z such that $D(A^\infty) \subset W \hookrightarrow Y$ and $D(A^\infty) \subset Z \hookrightarrow X$.

Proof. To obtain the claimed representation, take $\mathbb{x} := y \in \partial X = \partial\mathbb{X}$. By definition the Dirichlet operator $D_\lambda^{\mathbb{A},\mathbb{R}}$ maps $\mathbb{X}$ into the unique vector

$$\mathbb{u} := \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in \mathbb{X} \qquad \text{such that} \qquad \begin{cases} \mathbb{A}\mathbb{u} &= \lambda \mathbb{u}, \\ \mathbb{R}\mathbb{u} &= \mathbb{x}, \end{cases} \qquad \text{or rather} \qquad \begin{cases} v &= \lambda u, \\ Au &= \lambda v, \\ Lu &= \lambda x, \\ Ru &= y. \end{cases}$$

Thus, $u = D_{\lambda^2}^{A,R} y$, and (3.4) follows. Further, $D_\lambda^{\mathbb{A},\mathbb{R}} \in \mathcal{L}(\partial\mathbb{X}, \mathbb{Z})$ for every Banach space $\mathbb{Z}$ such that $D(\mathbb{A}^\infty) \subset \mathbb{Z} \hookrightarrow \mathbb{X}$. By Lemma 3.5 the claim follows. $\square$

We emphasize that the Dirichlet operators $D_\lambda^{\mathbb{A},\mathbb{R}}$, $\lambda \in \rho(\mathbb{A}_0)$, are compact if and only if ∂X is finite dimensional.

We now introduce a family of operators that will play an important role in the following. By Lemma 3.8 and (3.1), we obtain the following.

Lemma 3.9. *Let $\lambda \in \rho(\mathbb{A}_0)$. Then the operator*

$$\mathbb{B}_\lambda := \tilde{\mathbb{B}} + \mathbb{B} D_\lambda^{\mathbb{A},\mathbb{R}} \tag{3.5}$$

exists, is represented by

$$\mathbb{B}_\lambda = B_1 D_{\lambda^2}^{A,R} + \left(\frac{1}{\lambda} B_3 + B_4 \right) L D_{\lambda^2}^{A,R},$$

and is bounded on ∂X.

Using the family $(\mathbb{B}_\lambda)_{\lambda \in \rho(\mathbb{A}_0)}$ we can now perform a useful factorization, similar to those discussed in [8, § 2]. This will allow us to investigate the spectral properties of the matrix $\mathcal{A}$.

Lemma 3.10. *Let $\lambda \in \rho(\mathbb{A}_0)$. Then the factorization*

$$\lambda - \mathcal{A} = \mathcal{L}_\lambda \mathcal{A}_\lambda \mathcal{M}_\lambda := \begin{pmatrix} I_{\mathbb{X}} & 0 \\ -\mathbb{B} R(\lambda, \mathbb{A}_0) & I_{\partial\mathbb{X}} \end{pmatrix} \begin{pmatrix} \lambda - \mathbb{A}_0 & 0 \\ 0 & \lambda - \mathbb{B}_\lambda \end{pmatrix} \begin{pmatrix} I_{\mathbb{X}} & -D_\lambda^{\mathbb{A},\mathbb{R}} \\ 0 & I_{\partial\mathbb{X}} \end{pmatrix} \tag{3.6}$$

holds, and for all $\mu \in \mathbf{C}$ we further have

$$\mu - \mathcal{A} = \mathcal{L}_\lambda \begin{pmatrix} \mu - \mathbb{A}_0 & 0 \\ 0 & \mu - \mathbb{B}_\lambda \end{pmatrix} \mathcal{M}_\lambda + (\mu - \lambda) \begin{pmatrix} 0 & D_\lambda^{\mathbb{A},\mathbb{R}} \\ \mathbb{B} R(\lambda, \mathbb{A}_0) & -\mathbb{B} R(\lambda, \mathbb{A}_0) D_\lambda^{\mathbb{A},\mathbb{R}} \end{pmatrix}. \tag{3.7}$$

Proof. Let $\lambda \in \rho(\mathbb{A}_0)$ and take $\mathcal{U} := \binom{\mathbb{u}}{\mathbb{v}} \in \mathcal{X}$. Observe first that $\mathcal{U}$ is in the domain of the operator matrix $\mathcal{L}_\lambda \mathcal{A}_\lambda \mathcal{M}_\lambda$ if and only if $\mathbb{u} - D_\lambda^{\mathbb{A},\mathbb{R}} \mathbb{v} \in D(\mathbb{A}_0)$, that is, if and only if $\mathbb{R}\left(\mathbb{u} - D_\lambda^{\mathbb{A},\mathbb{R}}\mathbb{v}\right) = \mathbb{R}\mathbb{u} - \mathbb{v} = 0$. This shows that the domains of the operators in (3.6) agree. Moreover, we obtain

$$\begin{aligned}
&\begin{pmatrix} I_{\mathbb{X}} & 0 \\ -\mathbb{B}R(\lambda,\mathbb{A}_0) & I_{\partial\mathbb{X}} \end{pmatrix} \begin{pmatrix} \lambda - \mathbb{A}_0 & 0 \\ 0 & \lambda - \mathbb{B}_\lambda \end{pmatrix} \begin{pmatrix} I_{\mathbb{X}} & -D_\lambda^{\mathbb{A},\mathbb{R}} \\ 0 & I_{\partial\mathbb{X}} \end{pmatrix} \begin{pmatrix} \mathbb{u} \\ \mathbb{v} \end{pmatrix} \\
&\qquad = \begin{pmatrix} I_{\mathbb{X}} & 0 \\ -\mathbb{B}R(\lambda,\mathbb{A}_0) & I_{\partial\mathbb{X}} \end{pmatrix} \begin{pmatrix} \lambda - \mathbb{A}_0 & 0 \\ 0 & \lambda - \mathbb{B}_\lambda \end{pmatrix} \begin{pmatrix} \mathbb{u} - D_\lambda^{\mathbb{A},\mathbb{R}}\mathbb{v} \\ \mathbb{v} \end{pmatrix} \\
&\qquad = \begin{pmatrix} I_{\mathbb{X}} & 0 \\ -\mathbb{B}R(\lambda,\mathbb{A}_0) & I_{\partial\mathbb{X}} \end{pmatrix} \begin{pmatrix} (\lambda - \mathbb{A}_0)(\mathbb{u} - D_\lambda^{\mathbb{A},\mathbb{R}}\mathbb{v}) \\ \lambda\mathbb{v} - \mathbb{B}_\lambda \mathbb{v} \end{pmatrix} \\
&\qquad = \begin{pmatrix} (\lambda - \mathbb{A})(u - D_\lambda^{\mathbb{A},\mathbb{R}}\mathbb{v}) \\ -\mathbb{B}\mathbb{u} + \mathbb{B}D_\lambda^{\mathbb{A},\mathbb{R}}\mathbb{v} + \lambda\mathbb{v} - \mathbb{B}_\lambda\mathbb{v} \end{pmatrix} = \begin{pmatrix} \lambda - \mathbb{A} & 0 \\ -\mathbb{B} & \lambda - \tilde{\mathbb{B}} \end{pmatrix} \begin{pmatrix} \mathbb{u} \\ \mathbb{v} \end{pmatrix}
\end{aligned}$$

where we have used (3.5) and the fact that $D_\lambda^{\mathbb{A},\mathbb{R}}$ maps $\partial\mathbb{X}$ into $\ker(\lambda - \mathbb{A})$, by definition.

To show (3.7), take $\mu \in \mathbf{C}$ and observe that

$$\begin{aligned}
\mu - \mathcal{A} &= (\mu - \lambda)I_{\mathcal{X}} + \mathcal{L}_\lambda \mathcal{A}_\lambda \mathcal{M}_\lambda = (\mu - \lambda)I_{\mathcal{X}} + \mathcal{L}_\lambda \left[\begin{pmatrix} \mu - \mathbb{A}_0 & 0 \\ 0 & \mu - \mathbb{B}_\lambda \end{pmatrix} - (\mu - \lambda)I_{\mathcal{X}} \right] \mathcal{M}_\lambda \\
&= \mathcal{L}_\lambda \begin{pmatrix} \mu - \mathbb{A}_0 & 0 \\ 0 & \mu - \mathbb{B}_\lambda \end{pmatrix} \mathcal{M}_\lambda + (\mu - \lambda)(I_{\mathcal{X}} - \mathcal{L}_\lambda \mathcal{M}_\lambda).
\end{aligned}$$

One can check that

$$\mathcal{L}_\lambda \mathcal{M}_\lambda = \begin{pmatrix} I_{\mathbb{X}} & -D_\lambda^{\mathbb{A},\mathbb{R}} \\ -\mathbb{B}R(\lambda,\mathbb{A}_0) & I_{\partial\mathbb{X}} + \mathbb{B}R(\lambda,\mathbb{A}_0)D_\lambda^{\mathbb{A},\mathbb{R}} \end{pmatrix}, \tag{3.8}$$

and the claim follows. ☐

In many concrete cases, the spectrum of A_0, and hence by Lemma 3.4 of $\mathbb{A}_0$ are well-known. Hence it is interesting to decide whether a given $\lambda \in \rho(\mathbb{A}_0)$ is a spectral value of the larger matrix $\mathcal{A}$. Using Lemma 3.10, we can now derive a partial characterization whose main feature is the following: The spectrum and the point spectrum (denoted by σ and $P\sigma$, respectively) of a 4×4 operator matrix on $Y \times X \times \partial X \times \partial X$ is characterized by means of the operator pencils $(\mathbb{B}_\lambda)_{\lambda \in \rho(\mathbb{A}_0)}$ on ∂X.

Proposition 3.11. *For $\lambda \in \rho(\mathbb{A}_0)$ the equivalences*

$$\lambda \in \sigma(\mathcal{A}) \iff \lambda \in \sigma(\mathbb{B}_\lambda) \qquad \text{and} \qquad \lambda \in P\sigma(\mathcal{A}) \iff \lambda \in P\sigma(\mathbb{B}_\lambda) \tag{3.9}$$

hold. The set $\Gamma := \{\lambda \in \mathbf{C} : \lambda \in \rho(\mathbb{A}_0) \cap \rho(\mathbb{B}_\lambda)\} \subset \rho(\mathcal{A})$ is nonempty, and for $\lambda \in \Gamma$ the resolvent operator of $\mathcal{A}$ is given by

$$R(\lambda,\mathcal{A}) = \begin{pmatrix} R(\lambda,\mathbb{A}_0) + D_\lambda^{\mathbb{A},\mathbb{R}} R(\lambda,\mathbb{B}_\lambda)\mathbb{B}R(\lambda,\mathbb{A}_0) & D_\lambda^{\mathbb{A},\mathbb{R}} R(\lambda,\mathbb{B}_\lambda) \\ R(\lambda,\mathbb{B}_\lambda)\mathbb{B}R(\lambda,\mathbb{A}_0) & R(\lambda,\mathbb{B}_\lambda) \end{pmatrix}, \tag{3.10}$$

where the entries are as in Lemmas 3.4, 3.7, and 3.8.

Proof. Let $\lambda \in \rho(\mathbb{A}_0)$. Then the factorization (3.6) holds. Observe that the operators $\mathcal{L}_\lambda$, $\mathcal{M}_\lambda$ are bounded and invertible, hence $\lambda - \mathcal{A}$ is invertible if and only if the diagonal matrix $\mathcal{A}_\lambda$ is. We conclude that $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda \in \sigma(\mathbb{B}_\lambda)$. The latter equivalence in (3.9) follows likewise.

By Lemma 2.3(i) and Proposition 2.4, both the operators $\mathbb{A}_0$ and $\mathcal{A}$ are generators. Hence, their spectral bounds $s(\mathbb{A}_0), s(\mathcal{A}) < \infty$. To show that $\Gamma \neq \emptyset$, take thus $\lambda \geq \max\{s(\mathbb{A}_0), s(\mathcal{A})\}$ and deduce by (3.9) that $\lambda \in \Gamma$.

Finally, taking again into account (3.6) we obtain that for $\lambda \in \Gamma$ there holds $R(\lambda, \mathcal{A}) = \mathcal{M}_\lambda^{-1}\mathcal{A}_\lambda^{-1}\mathcal{L}_\lambda^{-1}$. A direct computation now yields (3.10). ☐

Moreover, neglecting the trivial case of finite dimensional X, the formula (3.10) allows us to obtain the following.

Theorem 3.12. *Let* dim $X = \infty$. *Then the following assertions are equivalent.*
a) $\mathcal{A}$ *has compact resolvent.*
b) $\mathbb{A}_0$ *has compact resolvent.*
c) ∂X *is finite dimensional and the embeddings* $[D(A_0)] \hookrightarrow Y$ *and* $Y \hookrightarrow X$ *are compact.*

Proof. Take $\lambda \in \Gamma$ as defined in Proposition 3.11, so that the resolvent operator $R(\lambda, \mathcal{A})$ is given by the formula (3.10).

a) $\Rightarrow$ *b)* Let $\mathcal{A}$ have compact resolvent. To begin with, the lower-right entry of $R(\lambda, \mathcal{A})$ is the resolvent operator of a bounded operator on ∂X, thus it is compact if and only if dim $\partial\mathbb{X} < \infty$. Therefore, the operator $D_\lambda^{\mathbb{A},\mathbb{R}} R(\lambda, \mathbb{B}_\lambda)\mathbb{B}R(\lambda, \mathbb{A}_0)$ is compact. Further, by assumption the upper left entry of $R(\lambda, \mathcal{A})$, i.e., $R(\lambda, \mathbb{A}_0) + D_\lambda^{\mathbb{A},\mathbb{R}} R(\lambda, \mathbb{B}_\lambda)\mathbb{B}R(\lambda, \mathbb{A}_0)$, is compact. It follows that their difference $R(\lambda, \mathbb{A}_0)$ is compact.

b) $\Rightarrow$ *a)* Let now $\mathbb{A}_0$ have compact resolvent. By Lemma 3.4, the identity on ∂X is compact, and this implies that dim $\partial X < \infty$. The claim now follows because the remaining entries of $R(\lambda, \mathcal{A})$ are bounded operators with finite-dimensional range.

b) $\Rightarrow$ *c)* In the following we consider $R(\lambda^2, A_0)$ as a bounded, non-compact operator from X to $[D(A_0)]$, and denote by i_Y and i_X the embeddings $[D(A_0)] \hookrightarrow Y$ and $Y \hookrightarrow X$, respectively. Let $\mathbb{A}_0$ have compact resolvent. We have already seen that necessarily dim $\partial X < \infty$. Moreover, the $(1,1)$-entry of $R(\lambda, \mathbb{A}_0)$, i.e., $i_Y \circ R(\lambda^2, A_0) \circ i_X$ is compact. Since also the $(2,1)$-entry $A_0 R(\lambda^2, A_0) \circ i_X = \lambda^2 i_X \circ i_Y R(\lambda^2, A_0) \circ i_X - i_X$ is compact, it follows that i_X is compact. Likewise, using the compactness of the (1,2)-entry of $R(\lambda, \mathbb{A}_0)$, we can show the compactness of i_Y, and the claim is proven.

c) $\Rightarrow$ *b)* Finally, let the embeddings $[D(A_0)] \hookrightarrow Y$ and $Y \hookrightarrow X$ be compact, and dim $\partial X < \infty$. Then the embedding $[D(\mathbb{A}_0)] \hookrightarrow \mathbb{X}$ is compact, and therefore $\mathbb{A}_0$ has compact resolvent. ☐

We now consider again the initial value problem associated with (ABC) and prove a conjecture formulated in [12, § 5]. Prof. J. Goldstein has informed us that his student C. Gal has recently obtained, by different methods, similar results on well-posedness and compactness issues. Gal's results have been obtained simultaneously to, but independently of ours; they will appear in [11].

Corollary 3.13. *Let the domain* Ω *be bounded. The matrix* $\mathcal{A}$ *associated with the abstract version of* (ABC) *on* $\Omega \subset \mathbb{R}^n$ *has compact resolvent if and only if* $n = 1$.

Proof. Recall that the embeddings $H^2(0,1) \hookrightarrow H^1(0,1) \hookrightarrow L^2(0,1)$ are compact. Then Theorem 3.12 yields the claim. ☐

To conclude this section, we mention that sharp results about the essential spectrum σ_{ess} of the operator matrix arising from the initial-boundary value problem associated with (ABC) have been obtained in [2, § 3]. The proofs therein are very technical, and only work if the domain Ω is bounded.

The formula (3.7) can however be used to obtain some results about $\sigma_{\text{ess}}(\mathcal{A})$, too. Our Propositions 3.14 and 4.1 below complement the results due to Beale. In particular, Proposition 3.14 also applies if we consider the motivating equation (ABC) to take place on the unbounded domain $\Omega = \mathbf{R}_+$.

Proposition 3.14. *Let ∂X be finite dimensional. Then the essential spectrum of $\mathcal{A}$ is given by*

$$\sigma_{\rm ess}(\mathcal{A}) = \sigma_{\rm ess}(\mathbb{A}_0),$$

and for the Fredholm index we have

$$ind(\mathcal{A} - \mu) = ind(\mathbb{A}_0 - \mu) \qquad \text{for all } \mu \notin \sigma_{\rm ess}(\mathbb{A}_0).$$

Proof. To begin with, we recall that the essential spectrum does neither change under compact additive perturbations, nor under similarity transformations (i.e., bounded invertible multiplicative perturbations).

Fix $\lambda \in \rho(\mathbb{A}_0)$, take into account (3.8), and observe that $I_{\mathcal{X}} - \mathcal{L}_\lambda \mathcal{M}_\lambda$ is a compact operator on $\mathcal{X}$. Moreover, $\mathcal{L}_\lambda$, $\mathcal{M}_\lambda$ are bounded and invertible. Thus, to decide whether a given $\mu \in \mathbf{C}$ is in the essential spectrum of $\mathcal{A}$, by (3.7) it suffices to check whether 0 is in the essential spectrum of the operator matrix

$$\begin{pmatrix} \mu - \mathbb{A}_0 & 0 \\ 0 & \mu - \mathbb{B}_\lambda \end{pmatrix} = \begin{pmatrix} \mu - \mathbb{A}_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mu - \mathbb{B}_\lambda \end{pmatrix}.$$

The second addend is a bounded operator with finite dimensional range, hence it does not affect the essential spectrum of the operator matrix on the left-hand side, and the claim follows. $\square$

4. The special case of $B_3 = 0$

After setting $y = \dot{x}$, (AIBVP$_2$) can equivalently be written as the second order problem with *integro-differential boundary conditions*

$$\begin{cases} \ddot{u}(t) &= Au(t), & t \in \mathbf{R}, \\ \dot{y}(t) &= B_1 u(t) + B_2 \dot{u}(t) + B_3 \left(h + \int_0^t y(s)\, ds \right) + B_4 y(t), & t \in \mathbf{R}, \\ y(t) &= Lu(t), & t \in \mathbf{R}, \\ u(0) &= f, \qquad \dot{u}(0) = g, \\ y(0) &= j. \end{cases}$$

In the special case of $B_3 = 0$, which we assume throughout this section, the initial value $x(0) = h$ is therefore superfluous, and we obtain an abstract second order problem with first order dynamical boundary conditions. Similar problems have been discussed, among others, in [6], and in fact some well-posedness result therein, cf. [6, Thm. 2.2], can be interpreted as a corollary of our Theorem 2.7. Moreover, observe that we can now replace $\mathbb{X} = Y \times X \times \partial X$ by $\mathbb{X} = Y \times X$, and the operator matrix $\mathbb{A}$ as defined in (3.1) by

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \qquad D(\mathbb{A}) = D(A) \times Y. \tag{4.1}$$

Accordingly, the operators $\mathbb{R}$ and $\mathbb{B}$ become

$$\mathbb{R} = \begin{pmatrix} R & 0 \end{pmatrix}, \qquad D(\mathbb{R}) = D(A) \times X, \qquad\qquad \mathbb{B} = \begin{pmatrix} B_1 + B_4 B_2 & 0 \end{pmatrix}, \qquad D(\mathbb{B}) = \mathbb{X}.$$

Then the operator matrix $\mathcal{A}$ defined in (2.8) becomes

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ A & 0 & 0 \\ B_1 + B_4 B_2 & 0 & B_4 \end{pmatrix}, \qquad D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ v \\ y \end{pmatrix} \in D(A) \times Y \times \partial X \,:\, Lu = B_2 u \right\}.$$

The main difference with the general setting of Section 3 is that the resolvent of $\mathbb{A}_0$ as well as the Dirichlet operators associated with $\mathbb{A}$ and $\mathbb{R}$ can be compact also in the case of dim $\partial X = \infty$.

This allows us to obtain the following.

Proposition 4.1. *Assume that there exists a Banach space Z containing $D(A)$ and compactly embedded in Y, and that also the embedding of Y in X is compact. Then*

$$\sigma_{\mathrm{ess}}(\mathcal{A})=\sigma_{\mathrm{ess}}(B_4) \qquad \text{and} \qquad ind(\mathcal{A}-\lambda)=ind(B_4-\lambda) \quad \text{for } \lambda\notin\sigma_{\mathrm{ess}}(B_4).$$

In particular, $\sigma_{\mathrm{ess}}(\mathcal{A})=\emptyset$ if and only if ∂X is finite dimensional.

Proof. Fix $\lambda\in\rho(\mathbb{A}_0)$. Then for all $\mu\in\mathbf{C}$ the factorization (3.7) holds. Taking into account Remark 3.3(b), we see that by assumption $D_\lambda^{\mathbb{A},\mathbb{R}}$ and $\mathbb{B}R(\lambda,\mathbb{A}_0)$ are compact operators from $\mathbb{X}$ to $\partial\mathbb{X}$ and from $\partial\mathbb{X}$ to $\mathbb{X}$, respectively. Thus, reasoning as in the proof of Proposition 3.14, we obtain that $\mu\in\sigma_{\mathrm{ess}}(\mathcal{A})$ if and only if $\mu\in\sigma_{\mathrm{ess}}(B_4)$. Here we have used the fact that $\sigma_{\mathrm{ess}}\big(\mathbb{B}_\lambda\big)=\sigma_{\mathrm{ess}}(\tilde{\mathbb{B}})=\sigma_{\mathrm{ess}}(B_4)$, and that $D(\mathbb{A}_0)$ is compactly embedded in $\mathbb{X}$, i.e., $\mathbb{A}_0$ has empty essential spectrum.

Finally, recall that a bounded operator has empty essential spectrum if and only if it acts on a finite dimensional space, cf. [9, § IV.1.20]. ☐

Example 4.2. In the context of our motivating equation (ABC), the assumption $B_3=0$ means that $k\equiv0$, hence the initial-boundary value problem becomes

$$(*)\qquad\left\{\begin{array}{rcll}\ddot\phi(t,x) &=& c^2\Delta\phi(t,x), & t\in\mathbf{R},\ x\in\Omega,\\ \ddot\delta(t,z) &=& -\frac{d(z)}{m(z)}\dot\delta(t,z)-\frac{\rho(z)}{m(z)}\dot\phi(t,z), & t\in\mathbf{R},\ z\in\partial\Omega,\\ \dot\delta(t,z) &=& \frac{\partial\phi}{\partial\nu}(t,z), & t\in\mathbf{R},\ z\in\partial\Omega,\\ \phi(0,\cdot) &=& f, \qquad \dot\phi(0,\cdot)=g, & \\ \dot\delta(0,\cdot) &=& j, & \end{array}\right.$$

on a bounded open domain $\Omega\subset\mathbf{R}^n$. Observe that $D(A)\subset Z:=H^{\frac32}(\Omega)$, and for $Y=H^1(\Omega)$ the embeddings $H^{\frac32}(\Omega)\hookrightarrow H^1(\Omega)\hookrightarrow L^2(\Omega)$ are compact by [17, Vol. I, Thm. 1.16.1]. Thus, by Proposition 4.1 the essential spectrum of $\mathcal{A}$ agrees with the essential spectrum of the bounded multiplication operator

$$(B_4u)(z)=-\frac{d(z)}{m(z)}u(z), \qquad u\in L^2(\partial\Omega),\ z\in\partial\Omega.$$

The essential spectrum of B_4 cannot be empty unless ∂X is finite dimensional, thus the essential spectrum of $\mathcal{A}$ cannot be empty unless $n=1$. ☐

Recall that in the context of our motivating equation (ABC) we always have $B_1=0$. However, if $B_3=0$ and the feedback B_1 is instead of the form $B_1=-B_4B_2$, then we obtain sharper spectral results. In fact, in this case $\mathcal{A}$ becomes a diagonal block matrix (with nondiagonal domain). Observe that the resolvent set of $\mathbb{A}_0$ is $\big\{\lambda\in\mathbf{C}:\lambda^2\in\rho(A_0)\big\}$, instead of $\big\{0\neq\lambda\in\mathbf{C}:\lambda^2\in\rho(A_0)\big\}$ as in Lemma 3.4, which is remarkable because in the context of our motivating equation (ABC) A_0 is the Laplacian with Robin boundary conditions, so that $0\in\rho(A_0)$ and therefore also $0\in\rho(\mathbb{A}_0)$.

We can now obtain an easier version of the equivalence (3.9) – getting rid of the operator pencil $\mathbb{B}_\lambda$, $\lambda\in\rho(\mathbb{A}_0)$ – and derive an alternative characterization of the spectrum of $\mathcal{A}$ that compliments the one already obtained in Proposition 3.11.

Corollary 4.3. *Let $B_3=0$ and $B_1=-B_4B_2$. Then the following hold.*
i) If $\lambda^2\in\rho(A_0)$, then

$$\lambda\in\sigma(\mathcal{A}) \iff \lambda\in\sigma(B_4) \qquad \text{and} \qquad \lambda\in P\sigma(\mathcal{A}) \iff \lambda\in P\sigma(B_4).$$

If $\lambda^2 \in \rho(A_0)$ and $\lambda \in \rho(B_4)$, then the resolvent operator $R(\lambda, \mathcal{A})$ is given by

$$R(\lambda, \mathcal{A}) := \begin{pmatrix} \lambda R(\lambda^2, A_0) & R(\lambda^2, A_0) & D_{\lambda^2}^{A,R} R(\lambda, B_4) \\ A_0 R(\lambda^2, A_0) & \lambda R(\lambda^2, A_0) & \lambda D_{\lambda^2}^{A,R} R(\lambda, B_4) \\ 0 & 0 & R(\lambda, B_4) \end{pmatrix}.$$

ii) If $\lambda \notin P\sigma(B_4)$, then

$$\lambda \in P\sigma(\mathcal{A}) \iff \lambda^2 \in P\sigma(A_0).$$

Proof. i) Lemma 3.9 yields that

$$\mathbb{B}_\lambda = (B_1 + B_4 L) D_{\lambda^2}^{A,R}, \qquad \lambda \in \rho(\mathbb{A}_0). \tag{4.2}$$

Now, taking into account (3.1) and Proposition 3.11 the claim follows.

ii) Let

$$(\mathcal{A} - \lambda)\mathcal{U} = \begin{pmatrix} v - \lambda u \\ Au - \lambda v \\ (B_4 - \lambda) R u \end{pmatrix} = 0.$$

Thus, we obtain that $(A_0 - \lambda^2)u = 0$ and the claim follows. □

Hence, we can sometimes obtain a complete characterization of the point spectrum.

Corollary 4.4. *Let $B_3 = 0$ and $B_1 = -B_4 B_2$. Assume that*

$$\{\lambda \in P\sigma(B_4) \,:\, \lambda^2 \in \sigma(A_0)\} = \emptyset. \tag{4.4}$$

Then

$$P\sigma(\mathcal{A}) = \{\lambda \in \mathbf{C} \,:\, \lambda^2 \in P\sigma(A_0) \ \text{ or } \ \lambda \in P\sigma(B_4)\}.$$

Observe that the condition (4.4) is in particular satisfied whenever A_0 is self-adjoint and invertible and B_4 has no eigenvalues on $i\mathbf{R} \setminus \{0\}$.

Example 4.5. Let $k \equiv 0$, and $B_1 = -B_4 B_2$, that is,

$$(B_1 f)(z) = -\frac{\rho(z) d(z)}{m^2(z)} f(z), \qquad f \in H^1(\Omega),\ z \in \partial\Omega.$$

Hence, we revisit Example 4.2 and consider a version of $(*)$ where we replace the second equation by

$$\ddot{\delta}(t,z) = -\frac{d(z)}{m(z)}\dot{\delta}(t,z) - \frac{\rho(z) d(z)}{m^2(z)}\phi(z) - \frac{\rho(z)}{m(z)}\dot{\phi}(t,z), \qquad t \in \mathbf{R},\ z \in \partial\Omega.$$

The Laplacian with Robin boundary conditions is self-adjoint and injective (with compact resolvent), thus the spectrum of A_0 consists of countably many strictly negative values diverging to $-\infty$, cf. [19, § IV.3–4]. On the other hand, B_4 is a multiplication operator. Thus, its spectrum agrees with the essential range of the function $-\frac{d(\cdot)}{m(\cdot)}$, while its point spectrum is given by $\big\{\lambda \in \mathbf{C} \,:\, \mu\{z \in \partial\Omega \,:\, d(z) + \lambda m(z) = 0\} > 0\big\}$, cf. [9, Ex. I.4.13(8)].

In particular, since by assumption m is real valued, the condition (4.4) is satisfied if the essential range of d does not contain any point on $i\mathbf{R}$. In this case by Corollary 4.3 we can completely characterize the point spectrum of the matrix $\mathcal{A}$ associated with $(*)$. In addition, $\mathcal{A}$ turns out to be invertible on $\mathcal{X}$. (Taking into account (4.2), one can see that this is not the case if $B_1 = 0$, because then $0 \in \rho(\mathcal{A})$ if and only if $0 \in \rho(B_4 L D_0^{A,R})$. This does not hold, since the normal derivative L vanishes on the set of constants.)

5. Neutral acoustic boundary conditions

Among the so-called *boundary contact problems* discussed by Belinsky in [4, § 3], the Timoschenko model

$$\text{(TM)} \qquad \left\{\begin{array}{rcll} \ddot{\phi}(t,x) &=& c^2\Delta\phi(t,x,z), & t\in\mathbf{R},\ x\in\Omega,\\ \frac{\partial\phi}{\partial\nu}(t,z) &=& 0, & t\in\mathbf{R},\ z\in\Gamma_0,\\ m(1-\Delta)\ddot{\delta}(t,z) &=& -d(z)\dot{\delta}(t,z)-k(z)\delta(t,z)-\rho\dot{\phi}(t,z), & t\in\mathbf{R},\ z\in\Gamma_1,\\ \dot{\delta}(t,z) &=& \frac{\partial\phi}{\partial\nu}(t,z), & t\in\mathbf{R},\ z\in\Gamma_1, \end{array}\right.$$

is particularly interesting, because it can be seen as a wave equation equipped with *neutral* acoustic boundary conditions. The aim of this section is to show how the methods introduced above can be applied to the present situation with minor changes.

For the geophysical explanation of this model we refer the reader to [4]. We only mention that the system (TM) models an ocean waveguide Ω covered (on the part Γ_1 of his surface $\partial\Omega$) by a thin pack ice layer with inertia of rotation. Belinsky investigates such a system for $\Omega\subset\mathbf{R}^2$ only and obtains some spectral properties.

Here the boundary $\partial\Omega$ is the disjoint union of Γ_0,Γ_1. Observe that, due to technical reasons, we consider the case of a medium of *homogeneous* density ρ filling a domain whose boundary has *homogeneous* mass m. However, we still allow k and d to be essentially bounded functions, whereas Belinsky assumes them to be constant.

In this section we cast such an equation into an abstract framework, and discuss its well-posedness – this is a new result to our knowledge.

To begin with, we introduce an operator M that will appear in the new neutral acoustic boundary conditions.

Assumption 5.1
(A_8) The operator $M : D(M)\subset\partial X\to\partial X$ is linear, closed, and satisfies $1\in\rho(M)$.

We can now consider the abstract second order initial-boundary value problem obtained by replacing the second equation in (AIBVP_2) by

$$\ddot{x}(t)-M\ddot{x}(t)=B_1u(t)+B_2\dot{u}(t)+B_3x(t)+B_4\dot{x}(t),\qquad t\in\mathbf{R}.$$

Thus, our aim is to show the well-posedness of the problem

$$(\mathrm{AIBVP}_2^\diamond) \qquad \left\{\begin{array}{rcll} \ddot{u}(t) &=& Au(t), & t\in\mathbf{R},\\ \ddot{x}(t) &=& B_1^\diamond u(t)+B_2^\diamond\dot{u}(t)+B_3^\diamond x(t)+B_4^\diamond\dot{x}(t), & t\in\mathbf{R},\\ \dot{x}(t) &=& Lu(t), & t\in\mathbf{R},\\ u(0) &=& f,\quad \dot{u}(0)=g, & \\ x(0) &=& h,\quad \dot{x}(0)=j, & \end{array}\right.$$

on X and ∂X, where

$$B_i^\diamond := R(1,M)B_i,\qquad i=1,2, \tag{5.1}$$

are bounded operators from Y to ∂X, and

$$B_i^\diamond := R(1,M)B_i,\qquad i=3,4, \tag{5.2}$$

are bounded operators on ∂X. Similarly, we consider the operator

$$R^\diamond := L-B_2^\diamond. \tag{5.3}$$

Observe now that, after replacing R by $R^\diamond$ and B_i by $B_i^\diamond$, $i=1,2,3,4$, all the corresponding Assumptions 2.1 are satisfied, except for (A_3) and (A_7). To fill this gap, we replace them by the following.

Assumptions 5.2
(A_3') The operator $R^\diamond : D(A) \to \partial X$ is linear and surjective.
(A_7') The restriction $A_0^\diamond := A_{|\ker R^\diamond}$ generates a cosine operator function with associated phase space $Y \times X$.

Under the assumptions (A_1), (A_2), (A_3'), (A_4)–(A_6), (A_7'), and (A_8) we promptly obtain the main result of this section.

Proposition 5.3. *The problem (AIBVP$_2^\diamond$) with abstract neutral acoustic boundary conditions is well-posed.*

Proof. Consider the operator matrix $\mathbb{A}$ as introduced in (2.2) and define the operators

$$\mathbb{R}^\diamond := \begin{pmatrix} R^\diamond & 0 & 0 \end{pmatrix}, \qquad D(\mathbb{R}^\diamond) := D(\mathbb{A}),$$
$$\mathbb{B}^\diamond := \begin{pmatrix} B_1^\diamond + B_4^\diamond B_2^\diamond & 0 & B_3^\diamond \end{pmatrix}, \qquad D(\mathbb{B}^\diamond) := \mathbb{X},$$
$$\tilde{\mathbb{B}}^\diamond := B_4^\diamond, \qquad D(\tilde{\mathbb{B}}^\diamond) := \partial\mathbb{X}.$$

We can now directly check that properties analogous to those in Lemma 2.3 are satisfied. Therefore, the well-posedness of

$$(\mathbb{AIBVP}^\diamond) \qquad \begin{cases} \dot{\mathbb{u}}(t) &=& \mathbb{A}\mathbb{u}(t), & t \in \mathbf{R}, \\ \dot{\mathbb{x}}(t) &=& \mathbb{B}^\diamond \mathbb{u}(t) + \tilde{\mathbb{B}}^\diamond \mathbb{x}(t), & t \in \mathbf{R}, \\ \mathbb{x}(t) &=& \mathbb{R}^\diamond \mathbb{u}(t), & t \in \mathbf{R}, \\ \mathbb{u}(0) &=& \mathbb{u}_0, \\ \mathbb{x}(0) &=& \mathbb{x}_0, \end{cases}$$

follows like in Proposition 2.4. Finally, reasoning like in Lemma 2.6 one obtains the equivalence between ($\mathbb{AIBVP}^\diamond$) and (AIBVP$_2^\diamond$), and the claim follows. □

It is usually not easy to check directly whether the assumption (A_3') is satisfied. Thus, we propose some conditions on interpolation inequalities that ensure its validity: In many concrete cases the restriction

$$A_L := A_{|\ker L}$$

generates an analytic semigroup on X, and we can therefore consider the interpolation spaces

$$X_\theta := [D(A_L), X]_{(1-\theta)}, \qquad 0 \le \theta \le 1,$$

cf. [18, Chapt. 1] for the abstract theory and [17, Vol. I, Chapt. 1] for concrete spaces.

Lemma 5.4. *Let L be surjective and A_L generate an analytic semigroup. Assume that for some $0 < \alpha < 1$ and $\tilde{\epsilon} \in (0, 1-\alpha)$ one has $D(A) \subset X_{\alpha+\tilde{\epsilon}}$ and $X_\alpha \hookrightarrow Y$. Then $R^\diamond$ is surjective.*

Proof. First of all we remark the existence of bounded Dirichlet operators $D_\lambda^{A,L}$ associated with A and L, $\lambda \in \rho(A_L)$. Further, we can reason as in [13, Lemma 2.4] and obtain that

$$\|D_\lambda^{A,L}\|_{\partial X, Y} = O(|\lambda|^{-\tilde{\epsilon}}) \qquad \text{for} \quad |\lambda| \to \infty, \ \mathrm{Re}(\lambda) > 0.$$

Thus, we can choose λ_0 large enough so that $\|B_2^\diamond D_{\lambda_0}^{A,L}\| < 1$, and consequently we can invert the operator $I_{\partial X} - B_2^\diamond D_{\lambda_0}^{A,L} = R^\diamond D_{\lambda_0}^{A,L}$. y We are now in the position to prove the surjectivity of $R^\diamond$. Take $x \in \partial X$, and observe that for the vector

$$u := D_{\lambda_0}^{A,L} \left(R^\diamond D_{\lambda_0}^{A,L} \right)^{-1} x$$

there holds $R^\diamond u = x$. □

We revisit the motivating example.

Theorem 5.5. *The initial value problem associated with the wave equation with neutral acoustic boundary conditions* (TM) *on an open bounded domain* $\Omega \subset \mathbf{R}^n$, $n \geq 1$, *with smooth boundary* $\partial\Omega = \Gamma_0 \cup \Gamma_1$ *is well-posed. In particular, for all initial data*

$$\phi(0,\cdot) \in H^2(\Omega), \qquad \dot{\phi}(0,\cdot) \in H^1(\Omega), \qquad \delta(0,\cdot) \in L^2(\Gamma_1),$$

$$\text{and} \qquad \dot{\delta}(0,\cdot) \in L^2(\Gamma_1) \qquad \text{such that} \qquad \frac{\partial \phi}{\partial \nu}(0,\cdot) = \dot{\delta}(0,\cdot)$$

there exists a classical solution on $(H^1(\Omega), L^2(\Omega), L^2(\Gamma_1))$ *continuously depending on them.*

Proof. We take over the setting introduced in Theorem 2.7 and adapt it to the current problem. We let

$$X := L^2(\Omega), \qquad Y := H^1(\Omega), \qquad \partial X := L^2(\Gamma_1).$$

Moreover, we set

$$A := c^2\Delta, \qquad D(A) := \left\{ u \in H^{\frac{3}{2}}(\Omega) \,:\, \Delta u \in L^2(\Omega),\ \frac{\partial u}{\partial \nu}_{|\Gamma_0} = 0 \right\},$$

$$(Rf)(z) = \frac{\partial f}{\partial \nu}(z) + \frac{\rho(z)}{m(z)} f(z), \qquad f \in D(R) = D(A),\ z \in \Gamma_1,$$

$$B_1 = 0, \qquad (B_2 f)(z) := -\frac{\rho(z)}{m(z)} f(z), \qquad f \in H^1(\Omega),\ z \in \Gamma_1,$$

$$(B_3 g)(z) := -\frac{k(z)}{m(z)} g(z), \qquad (B_4 g)(z) := -\frac{d(z)}{m(z)} g(z), \qquad g \in L^2(\Gamma_1),\ z \in \Gamma_1.$$

Further, we introduce the operator

$$M := \Delta, \qquad D(M) := H^2(\Gamma_1),$$

that is, the Laplace–Beltrami operator on Γ_1, and we can now define the auxiliary operators $B_i^\diamond$, $i = 1,2,3,4$, and $R^\diamond$ like in (5.1)–(5.3).

The operator M clearly satisfies the assumption (A_8), hence only the assumptions (A_3') and (A_7') still need to be checked.

To check (A_3'), first observe that $L = R + B_2$ is the normal derivative on Γ_1, which is surjective by [17, Vol. I, Thm. 2.7.4]. Moreover, consider the operator $A_L = A_{|\ker L}$ and observe that

$$D(A_L) = \left\{ u \in H^2(\Omega) \,:\, \frac{\partial u}{\partial \nu}_{|\partial\Omega} = 0 \right\},$$

that is, A_L turns out to be the Neumann Laplacian. Thus, A_L generates an analytic semigroup on X. Further, by [17, Vol. II, (4.14.32)], we obtain that

$$\big[[D(A_L)], L^2(\Omega)\big]_\theta = H^{2(1-\theta)}(\Omega) \qquad \text{for all } \theta \in \left(\frac{1}{4}, 1\right].$$

Therefore, for $X_\alpha = \big[[D(A_L)], L^2(\Omega)\big]_{1-\alpha}$, we obtain that

$$D(A) \subset H^{\frac{3}{2}}(\Omega) \subset X_{\alpha+\epsilon} \subset X_\alpha = H^1(\Omega) = Y \qquad \text{for } \alpha = \frac{1}{2} \text{ and all } 0 \leq \epsilon < \frac{1}{4}.$$

By Lemma 5.4 the assumption (A_3') is thus checked.

Finally, to check (A_7') we show that $A_0^\diamond$ is self-adjoint and dissipative. Recall that we are assuming ρ and m to be real constants, and observe that

$$D(A_0^\diamond) = \left\{ f \in H^2(\Omega) \,:\, \frac{\partial u}{\partial \nu}_{\mid \Gamma_0} = 0,\ \left(\frac{\partial u}{\partial \nu}_{\mid \Gamma_1} + \frac{\rho}{m} R(1,M)(u_{\mid \Gamma_1}) \right) = 0 \right\}.$$

Take $u, v \in D(A_0^\diamond)$ and obtain that

$$\begin{aligned} \left\langle A_0^\diamond u, v \right\rangle_X &= \int_\Omega \Delta u \cdot \overline{v}\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \overline{v}\, d\sigma - \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx \\ &= -\frac{\rho}{m} \int_{\Gamma_1} R(1,M)(u_{\mid \Gamma_1}) \cdot \overline{v}\, d\sigma - \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx. \end{aligned}$$

Taking into account the positivity and the self-adjointness of the operator $R(1, M)$, one obtains that $A_0^\diamond$ is self-adjoint and dissipative, hence it generates a cosine operator function. We still need to check that its phase space actually is $Y \times X$. To do so, it is convenient to use a variational argument. Integrating by parts one sees that $A_0^\diamond$ is *not* invertible, hence we need to consider its (invertible) perturbation $\tilde{A} := (A_0^\diamond - I_X)$, which also generates a cosine operator function. Observe that, due to the boundedness of the perturbation, $\tilde{A}$ and $A_0^\diamond$ have same form domain as well as same phase space. Reasoning as in the last lines of Example 2.8, the claim therefore follows if we can show that the form domain of $\tilde{A}$ is Y.

In fact, the sesquilinear form associated with $\tilde{A}$ is

$$a(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx + \frac{\rho}{m} \int_{\Gamma_1} R(1,M)^{\frac{1}{2}}(u_{\mid \Gamma_1}) \cdot \overline{R(1,M)^{\frac{1}{2}}(v_{\mid \Gamma_1})}\, d\sigma - \int_\Omega u \cdot \overline{v}\, dx,$$

whose domain is actually $H^1(\Omega) = Y$. $\square$

Acknowledgment. I gratefully thank Prof. Jerry Goldstein (University of Memphis) for bringing the theory of acoustic boundary conditions to my attention.

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